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THE EQUILIBRIUM SHAPES OF LENSES TAKING THE ACTUAL GRAVITATIONAL FIELD OF THE EARTH INTO ACCOUNT†

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A model formulation of the problem of the equilibrium shapes of a rotating oceanic lens of uniform density, the centre of which is at rest relative to the Earth is considered. All the components of the angular velocity of rotation of the Earth are taken into account, unlike, for example, in oceanography, where only one vertical component of this velocity is considered. The ocean surrounding the lens is assumed to be at rest, and its density is assumed to have a linear distribution. The equilibrium shape is the surface on which the pressures in the lens and in the ocean are equal, and here, on this surface, discontinuity of the tangential components of the velocities is permitted at points of both media. The exact solution of this problem, obtained earlier in [1] for the case of a uniform gravitational field of the Earth, is extended to the case of a potential gravitational field, approximating the actual field, taking into account of the variable nature of the centrifugal force field. The solution is approximate in nature and makes it possible, for example, to indicate the lower limit of the range of angular velocities of proper rotation of the lens, starting from which a more precise allowance for the gravitational field of the Earth is necessary, since it begins to have a considerable effect on the type of equilibrium shape sought. © 2004 Elsevier Ltd. All rights reserved.

In the past 15–20 years, at depths in the ocean of the order of 1 km, vast vortical formations have been found that differ from the surrounding water in temperature, salinity, density, chemical composition, and transparency, i.e. that contain an aqueous mass belonging in its hydrology to regions thousands of kilometres away from the points where the vortices are found. In shape, these formations resemble convex lenses, and they have been given this name. Oceanic lenses move thousands of kilometres, and they exist for several years. The mechanism responsible for their motion in the mass of the ocean (chiefly towards the south-west in the northern hemisphere) and the cause of their prolonged existence, of the order of several years, have drawn attention to this problem. The question of the longevity of the lenses is investigated below. So-called equilibrium shapes of lenses are plotted, taking into account the actual gravitational field of the Earth.

1. THE HYDRODYNAMIC EQUATIONS

We will consider a model formulation of the problem of the equilibrium shape of a rotating lens whose centre of mass is at rest relative to the Earth in the case where all components of the angular velocity of the Earth are taken into account (in oceanography, when considering problems of this kind, normally only the vertical component is taken into account).

We will select a system of coordinates $Cxyz$ with its origin at the centre of mass C of the lens and connected to the Earth. The x, y and z axes are directed eastward, northward, and along the outward local normal at the given point respectively. Let the centre of mass of the lens be at rest in the ocean. We will consider the complete equations (see, for example, [2]) of the steady motion of an ideal fluid in the body of a lens in Gromeka–Lamb form in the case of its constant density $\rho = \rho_0$

$$[(2\mathbf{\Omega} \times \operatorname{rot}\mathbf{v}) \times \mathbf{v}] + \operatorname{grad}\frac{v^2}{2} = -\frac{1}{\rho_0}\operatorname{grad}p + \mathbf{g} \tag{1.1}$$

where $\mathbf{v}$ is the vector of relative velocities of the lens particles, $\mathbf{\Omega}$ is the vector of the angular velocity of rotation of the Earth and p is the pressure. In the above dynamic equations, the Coriolis force of inertia (with a minus sign) is taken into account by the corresponding term on the left-hand side

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containing the vector $\mathbf{\Omega}$; the remaining terms correspond to the relative acceleration. The sum of the accelerations due to pure gravitation and the centrifugal force of inertia as a result of the rotation of the system *Cxyz* about the axis of rotation of the Earth corresponds to the vector **g** on the right-hand side. This vector is directed along the corresponding normal which, generally speaking, does not coincide in direction with the normal at the original of coordinates C. We will consider the gravitational field to be a potential field, characterized by a certain function U_{gr}.

In view of the choice of the axes *Cxyz*, the component $\Omega_x = 0$ and the projections Ω_y and Ω_z are latitude-dependent constants. Similar equations can also be written for the ocean, be replacing the constant density ρ_0 by a variable density ρ.

2. REPRESENTATION OF THE VECTOR g

Consider an arbitrary point on the ellipsoidal Earth corresponding to a geocentric latitude φ and a radius vector $\mathbf{R}_0$ from the geometric centre of the Earth. We will assume, for example, the pure gravitation is characterized in the vicinity of the given point by a constant modulus of acceleration g_0 = const and is directed strictly along a local radius vector towards the centre of the Earth. Then

$$g^2 = g_0^2 + \Omega^4 R_0^2 \cos^2\varphi - 2g_0\Omega^2 R_0 \cos^2\varphi$$

$$g_0 \sin\alpha = \Omega^2 R_0 \cos\varphi \sin\psi, \quad g\sin\alpha = \Omega^2 R_0 \cos\varphi \sin\varphi, \quad \psi = \alpha + \varphi$$

where α is the angle between the radius vector of the point considered and the local vertical. In particular, by virtue of this choice of angles, the components of the vector $\mathbf{\Omega}$ on the left-hand side of Eq. (1.1) have the form

$$\Omega_y = \Omega\cos\varphi, \quad \Omega_z = \Omega\sin\varphi$$

3. THE HYDROSTATICS OF THE OCEAN

From the equations of hydrostatics

$$g_x - \frac{1}{\rho}\frac{\partial p}{\partial x} = 0, \quad g_y - \frac{1}{\rho}\frac{\partial p}{\partial y} = 0, \quad g_z - \frac{1}{\rho}\frac{\partial p}{\partial z} = 0 \tag{3.1}$$

under conditions of a plane-parallel gravitational field of the Earth with vertical axis z it follows that

$$\partial p/\partial x = \partial p/\partial y = 0, \quad -\rho g_0 = \partial p/\partial z$$

where $g_0 \equiv$ const. This system has the solution $p(x, y, z) \equiv p(z)$ for any density $\rho = \rho(z)$. In the more complex problem being considered, the static solution is naturally sought in a similar way, by specifying ρ as a function of the local "height" h, which depends on x, y, z.

The question arises as to what the structure of the density of the ocean $\rho(x, y, z)$ should be in the general case in order for the function p always to exist. Along with the system of coordinates *Cxyz*, we will introduce into consideration the axes *OXYZ* connected to the centre of the Earth O. The OZ axis is directed towards the North Pole, and the OY axis lies in the meridional plane of point C and is the equatorial axis of the Earth. The OX axis is chosen such that the entire system *OXYZ* (like the system *Cxyz*) is right-handed. The formulae for the transition from one set of coordinates to another have the form

$$X = -x, \quad Y = R_0\cos\varphi - y\sin\psi + z\cos\psi, \quad Z = R_0\sin\varphi + y\cos\psi + z\sin\psi$$

The function $\rho(x, y, z)$ will be constructed in the vicinity of point C.

We will find the components of the overall accelerating **g** (due to gravitation on the centrifugal force) in axes connected to the Earth. We obtain

$$g_X = \frac{\partial U_{gr}}{\partial X} + \Omega^2 X, \quad g_Y = \frac{\partial U_{gr}}{\partial Y} + \Omega^2 Y, \quad g_Z = \frac{\partial U_{gr}}{\partial Z}$$

Introducing the 'general' potential function

$$U = U_{gr} + \Omega^2 r^2/2, \quad r^2 \equiv X^2 + Y^2 = R^2 - Z^2$$

we can write the components of the vector **g** as the gradient of the function U. We will seek a density $\rho(X, Y, Z)$ (or indeed $\rho = \rho(h)$, where $h = h(X, Y, Z)$) such that the conditions of congruence with respect to the function p in system (3.1) are satisfied, i.e.

$$\frac{\partial(g_X\rho)}{\partial Y} = \frac{\partial(g_Y\rho)}{\partial X}, \quad \frac{\partial(g_X\rho)}{\partial Z} = \frac{\partial(g_Z\rho)}{\partial X}, \quad \frac{\partial(g_Y\rho)}{\partial Z} = \frac{\partial(g_Z\rho)}{\partial Y} \tag{3.2}$$

We substitute into system (3.2) the vector function **g**, expressed in terms of the potential U. Assuming that the function U is twice continuously differentiable, we obtain the system of relations

$$\frac{\partial\rho}{\partial Y}\frac{\partial U}{\partial X} = \frac{\partial\rho}{\partial X}\frac{\partial U}{\partial Y}, \quad \frac{\partial\rho}{\partial Z}\frac{\partial U}{\partial X} = \frac{\partial\rho}{\partial X}\frac{\partial U}{\partial Z}, \quad \frac{\partial\rho}{\partial Z}\frac{\partial U}{\partial Y} = \frac{\partial\rho}{\partial Y}\frac{\partial U}{\partial Z} \tag{3.3}$$

in which the third relation is a consequence of the first two.

Consider the first equation of system (3.3). It solution with respect to ρ has the form

$$\rho(X, Y, Z) = \Phi_1(Z, U(X, Y, f(Z)))$$

(where Φ_1 and f are arbitrary functions), since the corresponding system

$$\frac{dX}{\partial U/\partial Y} = -\frac{dY}{\partial U/\partial X} = \frac{dZ}{0}$$

has two first integrals Z = const and $U(X, Y, f(Z))$ = const. In solving the second equation of system (3.3), we similarly obtain

$$\rho(X, Y, Z) = \Phi_2(Y, U(X, \varphi(Y), Z))$$

Comparing both of the expressions obtained for the same function ρ, we conclude that ρ may depend on Z and Y only in a complex manner by virtue of the dependence of the function Φ_1 and Φ_2 on U. On further consideration, it can be shown that $f(Z) \equiv Z$ and $\varphi(Y) \equiv Y$. From this it follows that the solution of system (3.3) is

$$\rho(X, Y, Z) = \rho(h(U)), \quad U = U(X, Y, Z)$$

where h is an arbitrary, fairly continuous function. The function U itself can be regarded as h, but it is clear that the function U has the dimension of acceleration multiplied by the linear dimension. Therefore, it is more convenient to regard h as having the dimension of length $h = -U/g_*$. Here g_* is the characteristic acceleration of the gravitational force, for example, $g_* = 9.81$ m/s^2. Thus, the density in a statically balanced ocean retains a constant value along equipotential surfaces of the overall force function and is equal to $\rho = \rho(-U/g_*)$.

We will find the pressure distribution in a statically balanced ocean. From the formulae of hydrostatics we have

$$\partial p/\partial X = \rho(-U/g_*)\partial U/\partial X = (-g_*)\partial(\int\rho(h)dh)/\partial X$$

and so on for the two other variables Y and Z. Hence

$$p \equiv p_f = p_0 - g_*\int\rho(h)dh$$

Assuming a linear density distribution in the vicinity of the characteristic level of the lenses h_0, we have $\rho = \rho_0 + kh_0 - kh$ (the notation $k = \rho_0 N^2/g_*$ is adopted, when N is the Brunt–Väisälä frequency at the level h_0), and hence

$$p_f = p_0 - g_*(\rho_0 + kh_0)h + g_*kh^2/2 = p_0 + (\rho_0 + kh_0)U + kU^2/(2g_*) =$$
$$= \text{const} + \rho_0 U + k(U - U_0)^2/(2g_*), \quad U_0 = -g_*h_0 \tag{3.4}$$

For example, substituting into this formula the quantity $U = -Rg_0 + \Omega^2 r^2/2$, which corresponds to the case of a gravitational field of constant modulus, we arrive at the expression

$$p_f = p_0 + (\rho_0 + kh_0)(\Omega^2 r^2/2 - Rg_0) - k(Rr^2\Omega^2/2 - g_0 R^2/2 - r^4\Omega^4/(8g_0))$$

For the approximate gravitational potential of an ellipsoidal Earth, the following representation is often used (see, for example, [3])

$$U_{\text{gr}} \approx \frac{\mu}{R}\left(1 + \frac{I_2 R_{\text{eq}}^2}{2R^2}\left(3\frac{Z^2}{R^2} - 1\right)\right) \tag{3.5}$$

where $\mu \approx 3.986 \times 10^{14}$ m^3/s^2, $I_2 \approx -1.082 \times 10^{-3}$ and R_{eq} is the equatorial radius of the Earth. From this we obtain the corresponding distribution of the pressure p_f in the ocean.

4. THE EQUILIBRIUM SHAPES OF LENSES

Let us consider the body of a lens. Here, the density is constant and is equal to ρ_0. According to relations (1.1), the pressure within the lens is the sum of dynamic and static terms. The "dynamic" term, if a "layered" [1] velocity field within the lens is used (corresponding to the case of a plane-parallel gravitational field, which differs little from the actual field), is described by the formula given in [1], whereas the static term, in view of the fact that there is no density gradient within the lens ($k = 0$), according to (3.4) is equal to

$$p = \rho_0 U + \text{const}$$

The solutions obtained enable us to formulate the hydrodynamic problem of the equilibrium shape of the lens. We will assume that the equilibrium shape of the lens is a surface, where the pressures on the side of the ocean and the lens are equal. At the same time, on this surface a jump in the tangential components of velocities is permitted [1]. As a result we have the following equation for the equilibrium surface.

$$p(0, 0, 0) + \frac{\rho_0\omega}{2}(\omega + 2\Omega_z)(x^2 + y^2) - 2\Omega_y\rho_0\omega yz + \frac{2\Omega_y^2\rho_0\omega z^2}{\omega + 2\Omega_z} = \text{const} + \frac{k}{2g_*}(U - U_0)^2 \tag{4.1}$$

The value of U_0 corresponds to the potential at the centre of the lens, point C, and ω is the angular velocity of proper rotation of the lens. When considering the "total" gravitational field, instead of U it is necessary to substitute the expression $U = U_{\text{gr}} + \Omega^2(X^2 + Y^2)/2$, what U_{gr} is defined by formula (3.5). When considering the approximate plane-parallel case $U = -g_* z$, where g_* is the acceleration due to gravity at the given latitude. In the case of the "total" field there are radicals on the right-hand side of Eq. (4.1); in the plane-parallel case it describes a second-order surface.

From an examination of the right-hand side of Eq. (4.1) it follows that, when the angular velocity ω is negligibly small, the equilibrium surface corresponds to the condition $U \approx U_0$ + const. This solution is of no practical interest. For example, in the case when $\Omega = 0$ and the gravitational field is of constant modulus, the solution is a spherical layer covering the World Ocean. On the other hand, when $|\omega| \to \infty$, Eq. (4.1) is converted into the equation of a circular cylinder $\omega(x^2 + y^2)$ = const. Naturally, other equilibrium shapes are possible, depending on the specific values of the parameters, including the parameters characterizing the deviation of the gravitational field from the plane-parallel field adopted in [1]. The ellipsoidal equilibrium shape found earlier [1] (for anticyclonically twisted lenses) corresponds to the following condition which limits the angular velocity of proper rotation: $\omega < 0$, $2\Omega\sin\varphi - |\omega| > 0$.

We will investigate the change in the equilibrium shape of a lens when the parameter ω changes in this range. The results given below were obtained for the condition of either a plane-parallel gravitational field [1] or a field of "central" form (3.5), and also on the assumption that $\omega = -\xi\Omega$, where the number $\xi \in (0, 1)$. The angle of latitude φ was taken to be 20°, and therefore it makes sense to examine the range $-0.684\Omega < \omega < 0$. An equilibrium shape similar to an ellipsoid of revolution about the vertical axis corresponds to the plane-parallel case; let the horizontal semi-axis of this be equal to 20 km; the size along the local vertical (it is similar to the length of the vertical semi-axis of the ellipsoid) is readily calculated in the case.

We will describe the characteristic deviations of the cross-sectional shapes at various levels z of the equilibrium surface obtained using the gravitational potential (3.5) (we will call such a surface "new") from the corresponding cross-sectional shapes of the "old" surface, corresponding to a plane-parallel field. Note that in the latter case the cross-sections are always circles of radius R_z depending on z, with centres displaced by an amount $y_0(z) = 2\Omega_y z/(\omega + 2\Omega_z)$ along the y axis. The values of the constants ρ_0, ω and Ω in formula (4.1) were taken to be the same in the two cases. The difference in pressures $p(0, 0, 0)$ – const at the centre of the lens and in the ocean at the level $z = 0$ was selected to be identical, i.e. such that in the case of a plane-parallel field the quantity $R_z(z = 0)$ is equal to 20 km. The square of the Brunt–Väisälä frequency was taken to be equal to $N^2 = 7 \times 10^{-6}\ \mathrm{s}^{-2}$.

As shown by calculations, the equilibrium shapes in the case of potential (3.5) are surfaces laying within the "old" equilibrium ellipsoid. The cross-sections of this surface at various z levels are smooth ovals lying within the "old" circles. The symbol Δ will be used to denote the distance between the "old" and "new" cross-sections at the corresponding z levels, measured from rays issuing from the centres of the "old" circles. At it turned out, the values of Δ possess extremal properties, corresponding to rays in the positive and negative directions of the y axis, and this gives the deviations Δ_{y+} and Δ_{y-}. By analogy, we also consider the quantities Δ_{x+} and Δ_{x-}, where, by virtue of a certain symmetry of the formulae, $\Delta_{x+} = \Delta_{x-} \equiv \Delta_x$.

The case $\omega = -0.2\Omega$. The given ratio of the angular velocities is similar to that actually observed. The maximum value of the z coordinate of the "old" ellipsoid (the vertical "dimension") is equal to $z_{\max} = 171.04$ m; the maximum magnitude of the displacements of the centres of the circles $y_{0\max} = 664.11$ m corresponds to this. The ratio of the major and minor semi-axes of the "old" ellipsoid amounts to 116.93.

The extremum values of the z coordinate of the "new" surface are similar in modulus and amount to $\approx\pm 168.36$ m. From an analysis of the numerical results, a number of conclusions can be drawn concerning the magnitudes of Δ. The greatest of the discrepancies Δ between the "old" and "new" surfaces is observed in the positive direction of the y axis, and here Δ_{y+} (curve *3* in Fig. 1) increases as $|z|$ increases. None of the quantities Δ_{y+}, Δ_{y-} and Δ_x differs greatly. Figure 1 also presents the even function $R_z(z)$ (curve *1*) corresponding to the "old" surface; the function $\Delta_{y+}(z)$ is almost even. Note that all quantities in Fig. 1 are given, albeit in different scales (along the horizontal) in metres. The greatest discrepancy between the points of the "old" and "new" surfaces of equilibrium obviously occurs in the plane $z = 0$ and amounts to about 1.5%.

Qualitatively, the form of the "old" and "new" surfaces (in the form of "cross-sections" along z) are shown in Fig. 2. The dashed curves show the contours of the "cross-section" of the equilibrium shape corresponding to the case of a plane-parallel gravitational field, and the solid curves show those

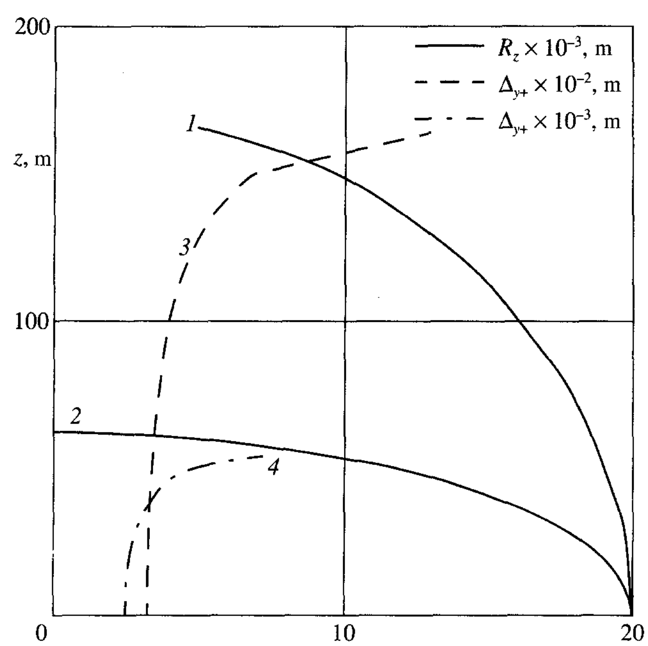

Fig. 1

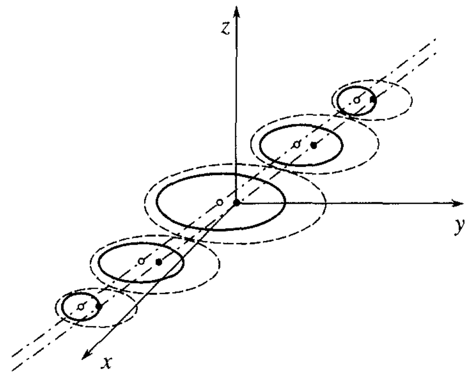

Fig. 2

corresponding to the case of the "total" field. The centres of the "old" circles lying on the above-mentioned line $y = y_0(z)$ are shown by the dark points. The centres of the "new" cross-sections are shown by the light points – they lie on an (almost) straight line parallel to $y = y_0(z)$, slightly displaced to the south. In the case in question this displacement amounts to about 12 m but, when the angular velocity ω is reduced further, it increases markedly. The bordering lines drawn through sections whose contours are given by the dashed and solid lines define the "old" and "new" surfaces of equilibrium. They are not shown in Fig. 2.

The case $\omega = -0.02\Omega$. The maximum vertical "dimension" of the "old" ellipsoid $z_{max} = 63.35$ m, to which the maximum magnitude of displacements of the centres of the circles $y_{0max} = 179.30$ m corresponds. The ratio of the major and minor semi-axes of the "old" ellipsoid is 315.70.

The extremum values of the z coordinate of the "new" surface practically coincide (in modulus), i.e. $z = 55.72$ m. The values of Δ_{y+} increase from 2487 m at $z = 0$ to 7181 m at $z = 55$ m. The graphs of Δ_{y+} and R_z as functions of z in this case are represented by curves *4* and *2* in Fig. 1.

The case $\omega = -0.005\Omega$. The maximum vertical dimension of the "old" ellipsoid $z_{max} \approx 32$ m, to which the maximum magnitude of the displacements of the centres of the circles $y_{0max} \approx 88$ m corresponds. The ratio of the major and minor semi-axes of the "old" ellipsoid is 624. The maximum vertical dimension of the lens corresponding to the "new" equilibrium shape amounts to about 15 m. A deviation $\Delta_{y+} \approx$ 13 528 m occurs at the level $z = 0$.

The above results indicate that, in the case when $\omega = -0.02\Omega$, the two shapes differ roughly by 10%, and, for an angular velocity reduced by a factor of 4, the effect of gravitational factors on the equilibrium shape can be decisive. It must be pointed out, however, the more accurate allowance for the effect of gravitational factors requires the constructing of a more accurate velocity field within the lens than that proposed earlier [1].

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